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BAER COTORSION PAIRS

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ABSTRACT

Let R be a unital associative ring and $\mathfrak{V}, \mathfrak{W}$ two classes of left R-modules. In [St3] the notion of a $(\mathfrak{V}, \mathfrak{W})$-cotorsion pair was introduced. In analogy to classical cotorsion pairs, a pair $(\mathcal{V}, \mathcal{W})$ of subclasses $\mathcal{V} \subseteq \mathfrak{V}$ and $\mathcal{W} \subseteq \mathfrak{W}$ is called a $(\mathfrak{V}, \mathfrak{W})$-cotorsion pair if it is maximal with respect to the classes $\mathfrak{V}, \mathfrak{W}$ and the condition $\mathrm{Ext}^1_R(V, W) = 0$ for all $V \in \mathcal{V}$ and $W \in \mathcal{W}$. In this paper we study $(\mathfrak{Tf}, \mathfrak{T})$-cotorsion pairs where $R = \mathbb{Z}$ and $\mathfrak{Tf}$ is the class of all torsion-free abelian groups and $\mathfrak{T}$ is the class of all torsion abelian groups. A complete characterization is obtained assuming $V = L$. For example, it is shown that every $(\mathfrak{Tf}, \mathfrak{T})$-cotorsion pair is singly cogenerated under $V = L$.

Introduction

Let R be any unital and associative ring and let $\mathfrak{V}$ and $\mathfrak{W}$ be two classes of left R-modules. In [St3] the author introduced the notion of $(\mathfrak{V}, \mathfrak{W})$-cotorsion pairs which is closely related to the well-known notion of cotorsion pairs (for abelian groups) which goes back to Salce [Sa]. In analogy to Dickson's notion of a torsion theory, a **cotorsion theory** or **cotorsion pair** is a pair $(\mathcal{A}, \mathcal{B})$ of classes of abelian groups which is maximal with respect to the property that $\mathrm{Ext}^1_{\mathbb{Z}}(A, B) = 0$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Cotorsion pairs form a complete lattice which has a complicated structure as was shown in [GöShWa]. In the obvious way this notion can be extended to any class of modules over any kind of ring. Since the late 1970's cotorsion pairs (over arbitrary rings) have been

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studied extensively by several mathematicians and recently led to the solution of the flat cover conjecture (see [Xu]), a major problem in module theory (see Bican–El Bashir–Enochs [BiBaEn]). Cotorsion pairs also play an important role in the theory of tilting and cotilting modules (see [Baz1], [Baz2], [Baz3], [BazEkTr], [BazSa], [EkShTr], [Tr1], [Tr2] and the references given there). In this context it has been of particular interest whether or not a given cotorsion pair is complete, i.e. has enough injectives and projectives.

In analogy to cotorsion pairs a $(\mathfrak{V}, \mathfrak{W})$-cotorsion pair is defined as a pair $(\mathcal{V}, \mathcal{W})$ of classes $\mathcal{V} \subseteq \mathfrak{V}$ and $\mathcal{W} \subseteq \mathfrak{W}$ which is maximal with respect to the condition that $\mathrm{Ext}^1_R(V, W) = 0$ for all $V \in \mathcal{V}$ and $W \in \mathcal{W}$ but only related to the two classes $\mathfrak{V}$ and $\mathfrak{W}$. This is a refinement of the classical notion of a cotorsion pair, i.e. $(\mathcal{V}, \mathcal{W})$ is a cotorsion pair if and only if it is a $(R\text{-}Mod, R\text{-}Mod)$-cotorsion pair.

In this paper we study the case when $R = \mathbb{Z}$ and $\mathfrak{V} = \mathfrak{Tf}$, $\mathfrak{W} = \mathfrak{T}$ where $\mathfrak{Tf}$ is the class of all torsion-free abelian groups and $\mathfrak{T}$ is the class of all torsion abelian groups. A $(\mathfrak{Tf}, \mathfrak{T})$-cotorsion pair is called a Baer cotorsion pair since the motivation comes from an old problem due to R. Baer. In [Ba1] Baer asked to characterize all pairs (G, T) of torsion-free abelian groups G and torsion abelian groups T satisfying $\mathrm{Ext}^1_{\mathbb{Z}}(G, T) = 0$. In particular cases the problem has been solved for torsion-free groups of size at most $\aleph_2$ or by requiring $\mathrm{Ext}^1_{\mathbb{Z}}(G, T) = 0$ for a proper class of T's (see e.g. [Ba1], [Gr], and also [EkFuSh]). Finally, the author solved it completely assuming $V = L$ in [St2]. Using these results we give a complete characterization of the $(\mathfrak{Tf}, \mathfrak{T})$-cotorsion pairs assuming Goedel's constructible universe $V = L$ and hence solve Baer's problem again but from the viewpoint of Baer cotorsion pairs assuming the additional set-theoretic assumption $V = L$.

We assume that the reader is familiar with basic homological algebra and the theory of abelian groups (see [EkMe] and [Fu1], [Fu2] for further details). In particular, we assume knowledge of the concept of height sequences (of elements $x \in G$ inside the abelian group G), types (isomorphism classes of rational groups $R \subseteq \mathbb{Q}$), and the lattice of types. Since there is no danger of confusion we shall always identify types with a rational group in the isomorphism class (see [Ma] for details on types). For further details on cotorsion pairs we refer the reader to the two books by Eklof–Mekler [EkMe] and Göbel–Trlifaj [GöTr3].

The results in this paper also appear in the author's Habilitationsschrift [St4].

1. $(\mathfrak{V},\mathfrak{W})$-cotorsion pairs

In this section we recall some definitions and results from [St3]. Throughout let R be any unital associative ring and $\mathfrak{V},\mathfrak{W}$ two classes of left R-modules. The following definition is a refinement of the notion of a classical cotorsion pair (see [St3]).

Definition 1.1: Let $\mathcal{V} \subseteq \mathfrak{V}$ be a subclass of $\mathfrak{V}$ and let $\mathcal{W} \subseteq \mathfrak{W}$ be a subclass of $\mathfrak{W}$. The pair $(\mathcal{V},\mathcal{W})$ is called a **$(\mathfrak{V},\mathfrak{W})$-cotorsion pair** if it satisfies the following three conditions:

(i) $\operatorname{Ext}^1_R(V,W) = 0$ for all $V \in \mathcal{V}$ and $W \in \mathcal{W}$;
(ii) if $X \in \mathfrak{W}$ and $\operatorname{Ext}^1_R(V,X) = 0$ for every $V \in \mathcal{V}$ then $X \in \mathcal{W}$;
(iii) if $Y \in \mathfrak{V}$ and $\operatorname{Ext}^1_R(Y,W) = 0$ for every $W \in \mathcal{W}$ then $Y \in \mathcal{V}$.

The class $\mathcal{V}$ is called the **$\mathfrak{V}$-cotorsion-free class** and the class $\mathcal{W}$ is called the **$\mathfrak{W}$-cotorsion class** of the $(\mathfrak{V},\mathfrak{W})$-cotorsion pair $(\mathcal{V},\mathcal{W})$.

In other words, the pair $(\mathcal{V},\mathcal{W})$ is a $(\mathfrak{V},\mathfrak{W})$-cotorsion pair if it is maximal with respect to the condition $\operatorname{Ext}^1_R(V,W) = 0$ for all $V \in \mathcal{V}$ and $W \in \mathcal{W}$ but only related to the two classes $\mathfrak{V}$ and $\mathfrak{W}$. Clearly, a classical cotorsion pair (as defined in [Sa]) is nothing else but a (R-Mod,R-Mod)-cotorsion pair. Moreover, if the pair $(\mathfrak{W},\mathfrak{V})$ forms a torsion theory, i.e. if $\operatorname{Hom}_R(W,V) = 0$ for all $W \in \mathfrak{W}$ and $V \in \mathfrak{V}$ (see Dickson [Di]), then a $(\mathfrak{V},\mathfrak{W})$-cotorsion pair was called a **torsion cotorsion pair** in [St3].

As for cotorsion pairs we may define the $(\mathfrak{V},\mathfrak{W})$-cotorsion pair generated (cogenerated) by a class of R-modules. To do so, we need a replacement of the perpendicular operations. Recall that, if $\mathcal{T}$ is a class of R-modules, then

$$\mathcal{T}^{\perp} = \{X \in R\text{-}Mod\colon \operatorname{Ext}^1_R(T,X) = 0 \text{ for all } T \in \mathcal{T}\}$$

and dually

$${}^{\perp}\mathcal{T} = \{X \in R\text{-}Mod\colon \operatorname{Ext}^1_R(X,T) = 0 \text{ for all } T \in \mathcal{T}\}.$$

In analogy we let $\mathcal{TC}_{\mathfrak{W}}(\mathcal{T}) = \mathcal{T}^{\perp} \cap \mathfrak{W}$ and $\mathcal{FC}_{\mathfrak{V}}(\mathcal{T}) = {}^{\perp}\mathcal{T} \cap \mathfrak{V}$.

Definition 1.2: Let $\mathcal{T}$ be a class of R-modules. We call

$$(\mathcal{FC}_{\mathfrak{V}}(\mathcal{T}), \mathcal{TC}_{\mathfrak{W}}(\mathcal{FC}_{\mathfrak{V}}(\mathcal{T})))$$

the $(\mathfrak{V},\mathfrak{W})$-cotorsion pair **generated** by $\mathcal{T}$ and dually

$$(\mathcal{FC}_{\mathfrak{V}}(\mathcal{TC}_{\mathfrak{W}}(\mathcal{T})), \mathcal{TC}_{\mathfrak{W}}(\mathcal{T}))$$

the $(\mathfrak{V},\mathfrak{W})$-cotorsion pair **cogenerated** by $\mathcal{T}$. If $\mathcal{T}$ consists of a single R-module then we speak of the **singly generated** or **singly cogenerated** $(\mathfrak{V},\mathfrak{W})$-cotorsion pair.

It is easy to check that the $(\mathfrak{V},\mathfrak{W})$-cotorsion pairs (co) generated by a class $\mathcal{T}$ are indeed cotorsion pairs (see also [St3]).

The class of all $(\mathfrak{V},\mathfrak{W})$-cotorsion pairs is partially ordered by the reverse inclusion in the first component: $(\mathcal{V},\mathcal{W}) \leq (\mathcal{V}',\mathcal{W}')$ if and only if $\mathcal{V} \supseteq \mathcal{V}'$ or, equivalently, $\mathcal{W} \subseteq \mathcal{W}'$. Moreover, the $(\mathfrak{V},\mathfrak{W})$-cotorsion pairs form a complete lattice: Given a sequence of $(\mathfrak{V},\mathfrak{W})$-cotorsion pairs $((\mathcal{V}_i,\mathcal{W}_i) : i \in I)$, the supremum is given by $(\bigcap_{i\in I}\mathcal{V}_i, \mathcal{TC}_{\mathfrak{W}}(\bigcap_{i\in I}\mathcal{V}_i))$ and the infimum by $(\mathcal{FC}_{\mathfrak{V}}(\bigcap_{i\in I}\mathcal{W}_i), \bigcap_{i\in I}\mathcal{W}_i)$.

Regarding the connection between cotorsion pairs and $(\mathfrak{V},\mathfrak{W})$-cotorsion pairs the following was shown in [St3].

LEMMA 1.3: *Let $(\mathcal{V},\mathcal{W})$ be a $(\mathfrak{V},\mathfrak{W})$-cotorsion pair. If $(\mathcal{A},\mathcal{B})$ is either the cotorsion pair generated by $\mathcal{W}$ or the cotorsion pair cogenerated by $\mathcal{V}$, then $(\mathcal{V},\mathcal{W}) = (\mathcal{A}\cap\mathfrak{V}, \mathcal{B}\cap\mathfrak{W})$.*

Thus every $(\mathfrak{V},\mathfrak{W})$-cotorsion pair $(\mathcal{V},\mathcal{W})$ is naturally induced by a cotorsion pair $(\mathcal{A},\mathcal{B})$ in the sense that $(\mathcal{V},\mathcal{W}) = (\mathcal{A}\cap\mathfrak{V}, \mathcal{B}\cap\mathfrak{W})$.

Finally, a list of problems was given in [St3] which should be solved for a particular triple $(R,\mathfrak{V},\mathfrak{W})$, in order to get insight in the structure of the $(\mathfrak{V},\mathfrak{W})$-cotorsion pairs. A module $X \in \mathcal{V}\cap\mathcal{W}$ is called a **splitter** of the $(\mathfrak{V},\mathfrak{W})$-cotorsion pair $(\mathcal{V},\mathcal{W})$. Moreover, the $(\mathfrak{V},\mathfrak{W})$-cotorsion pair $(\mathcal{V},\mathcal{W})$ has **enough injectives** if, for every $M \in \mathfrak{V}$, there exists an exact sequence

$$0 \to M \to W \to V \to 0$$

with $V \in \mathcal{V}$ and $W \in \mathcal{W}$. Such a short exact sequence is called a **special $\mathcal{W}$-preenvelope** of M. Note that for any hereditary ring R, the module $V \in \mathcal{V}\cap\mathcal{W}$ is a splitter. Dually, we say that $(\mathcal{V},\mathcal{W})$ has **enough projectives** if, for every $M \in \mathfrak{W}$, there exists an exact sequence

$$0 \to W \to V \to M \to 0$$

with $V \in \mathcal{V}$ and $W \in \mathcal{W}$. Such a short exact sequence is called a **special $\mathcal{V}$-precover** of M and again for hereditary rings, the module $W \in \mathcal{V}\cap\mathcal{W}$ is a splitter of the $(\mathfrak{V},\mathfrak{W})$-cotorsion pair $(\mathcal{V},\mathcal{W})$.

LIST OF PROBLEMS 1.4: *Let R be any reasonable ring and $\mathfrak{V}$, $\mathfrak{W}$ be two classes of R-modules which have nice closure properties. Solve the following problems:*

(i) *What are the splitters of a $(\mathfrak{V},\mathfrak{W})$-cotorsion pair?*
(ii) *Does a $(\mathfrak{V},\mathfrak{W})$-cotorsion pair have enough injectives or projectives respectively?*
(iii) *Characterize the singly generated or singly cogenerated $(\mathfrak{V},\mathfrak{W})$-cotorsion pairs respectively.*
(iv) *Determine the lattice of all $(\mathfrak{V},\mathfrak{W})$-cotorsion pairs; what is the minimal and maximal element?*
(v) *Can you develop some kind of approximation theory?*
(vi) *What information does a $(\mathfrak{V},\mathfrak{W})$-cotorsion pair $(\mathcal{V},\mathcal{W})$ give about the cotorsion pair generated by $\mathcal{W}$ or cogenerated by $\mathcal{V}$, respectively?*

2. The Baer cotorsion pairs

In this section we shall consider a particular torsion cotorsion pair which was motivated by a problem due to R. Baer who asked in [Ba1] for a characterization of all pairs (G,T) of torsion-free abelian groups G and torsion abelian groups T such that $\mathrm{Ext}^1_{\mathbb{Z}}(G,T)=0$. In the sequel all groups will be abelian, $\mathrm{Ext}(-,-)$ denotes $\mathrm{Ext}^1_{\mathbb{Z}}(-,-)$ and cotorsion pair means $(\mathfrak{Ab},\mathfrak{Ab})$-cotorsion pair where $\mathfrak{Ab}$ is the class of all abelian groups. Recall that $\mathfrak{Tf}$ is the class of all torsion-free abelian groups and $\mathfrak{T}$ the class of all torsion abelian groups. Trivially, the pair $(\mathfrak{T},\mathfrak{Tf})$ forms a torsion theory and hence we are interested in the torsion cotorsion pairs associated to this torsion theory, i.e. the $(\mathfrak{Tf},\mathfrak{T})$-cotorsion pairs.

Definition 2.1: A $(\mathfrak{Tf},\mathfrak{T})$-cotorsion pair is called a **Baer cotorsion pair**.

We shall give answers to most of the problems from the List of Problems 1.4. It is immediate to see that for every Baer cotorsion pair $(\mathcal{G},\mathcal{T})$ the class $\mathcal{G}$ contains the class $\mathfrak{F}$ of all free abelian groups and the class $\mathcal{T}$ contains all torsion cotorsion abelian groups, i.e. $\mathfrak{T}\cap\mathfrak{C}\subseteq\mathcal{T}$, where $\mathfrak{C}$ is the class of all cotorsion groups, i.e. abelian groups G such that $\mathrm{Ext}(\mathbb{Q},G)=0$ or equivalently $\mathrm{Ext}(H,G)=0$ for all $H\in\mathfrak{Tf}$. Moreover, $\mathcal{G}$ is closed under taking isomorphic copies, subgroups, arbitrary direct sums and extensions while $\mathcal{T}$ is closed under taking isomorphic copies, epimorphic images, extensions but not under direct products since a product of torsion groups need not be torsion. Furthermore, note that, in contrast to the classical cotorsion pairs, a Baer cotorsion pair never has enough injectives nor projectives and obviously does not contain any splitters except for the trivial group. Hence problems (i), (ii) and (v) of the List of Problems 1.4 are not of interest in this section.

Using Griffith's solution to the Baer splitting problem (see [Gr]) we have the following theorem.

THEOREM 2.2: *The Baer cotorsion pairs form a complete lattice with maximal element* $(\mathfrak{F}, \mathfrak{T})$ *and minimal element* $(\mathfrak{Tf}, \mathfrak{C} \cap \mathfrak{T})$.

Proof: By [St3, Theorem 1.6] the Baer cotorsion pairs form a complete lattice and clearly $(\mathfrak{F}, \mathfrak{T})$ and $(\mathfrak{Tf}, \mathfrak{C} \cap \mathfrak{T})$ are maximal and minimal respectively. Thus all we have to show is that $(\mathfrak{F}, \mathfrak{T})$ is a Baer cotorsion pair. But this is immediate since [Gr] shows that an abelian group G is free if and only if $\mathrm{Ext}(G,T) = 0$ for all torsion groups T. ∎

Before we continue we would like to give some examples of Baer cotorsion pairs different from the maximal and the minimal ones. A helpful result was proved in [StWa] by Wallutis and the author.

PROPOSITION 2.3: *Let R be a rational group with $\chi^R(1) = (r_p)_{p\in\Pi}$ and let $T = \bigoplus_{p\in\Pi} T_p$ be a reduced torsion group with p-components T_p.*

Then $\mathrm{Ext}(R,T) = 0$ if and only if the following conditions are satisfied:

(i) *T_p is bounded for all p such that $r_p = \infty$;*

(ii) *$T_p = 0$ for almost all p such that $r_p \neq 0$.*

EXAMPLE 2.4: *Let $\mathcal{H} = \bigcup_{p\in\Pi} \mathfrak{T}_p$ be the class of all p-groups for all primes p. Then the Baer cotorsion pair generated by $\mathcal{H}$ is different from the maximal and minimal Baer cotorsion pair.*

Proof: Let $(\mathcal{G}, \mathcal{T})$ be the Baer cotorsion pair generated by $\mathcal{H}$. By Proposition 2.3, the rational group $R = \langle 1/p : p \in \Pi\rangle$ is contained in $\mathcal{G}$ and thus $(\mathcal{G}, \mathcal{T})$ is different from the maximal Baer cotorsion pair. On the other hand, the rational group $R = \langle 1/p^n : n \in \omega\rangle$ is not contained in $\mathcal{G}$ for every prime p, by Proposition 2.3, and therefore $(\mathcal{G}, \mathcal{T})$ is also different from the minimal Baer cotorsion pair. ∎

EXAMPLE 2.5: *Let $\mathcal{H}_\lambda$ be the class of all groups $T \in \mathcal{H}$ such that $|T| \leq \lambda$ for some infinite cardinal λ. Then the Baer cotorsion pair generated by $\mathcal{H}_\lambda$ is generated by a set of torsion groups but is not singly generated.*

Proof: Let $(\mathcal{G}, \mathcal{T})$ be the Baer cotorsion pair generated by $\mathcal{H}_\lambda$. Obviously, $(\mathcal{G}, \mathcal{T})$ is generated by a set since $\mathcal{H}_\lambda$ is a set. As in Example 2.4 the rational group $R = \langle 1/p : p \in \Pi\rangle$ is contained in $\mathcal{G}$. Assume that $(\mathcal{G}, \mathcal{T})$ is generated by a single torsion group T. Then T must have infinitely many non-trivial p-components which implies $R \notin \mathcal{G}$ by Proposition 2.3 — a contradiction. ∎

EXAMPLE 2.6: *Let P be a proper subset of the natural primes Π. If $\mathfrak{T}_P = \bigoplus_{p \in P} \mathfrak{T}_p$ is the class of all groups $T \in \mathfrak{T}$ with trivial p-components for $p \notin P$, then the Baer cotorsion pair generated by $\mathfrak{T}_P$ is different from the maximal and minimal Baer cotorsion pairs.*

Proof: Follows as in the proof of Example 2.4 using the appropriate rational groups. ∎

We now clarify the connection between cotorsion pairs and Baer cotorsion pairs. The first lemma is a consequence of Lemma 1.3.

LEMMA 2.7: *Let $(\mathcal{G}, \mathcal{T})$ be a Baer cotorsion pair. If $(\mathcal{A}, \mathcal{B})$ is either the cotorsion pair generated by $\mathcal{T}$ or the cotorsion pair cogenerated by $\mathcal{G}$, then $(\mathcal{G}, \mathcal{T}) = (\mathcal{A} \cap \mathfrak{T}\mathfrak{f}, \mathcal{B} \cap \mathfrak{T}) = (\mathcal{A}, \mathcal{B} \cap \mathfrak{T})$.*

Proof: Lemma 1.3 shows that $(\mathcal{G}, \mathcal{T}) = (\mathcal{A} \cap \mathfrak{T}\mathfrak{f}, \mathcal{B} \cap \mathfrak{T})$ holds. The second equality now follows since $\mathcal{T}$ contains all torsion cotorsion groups and thus, in either case, the class $\mathcal{A}$ must consist of torsion-free abelian groups only. ∎

The above Lemma 2.7 shows that every Baer cotorsion pair $(\mathcal{G}, \mathcal{T})$ is induced by a cotorsion pair $(\mathcal{A}, \mathcal{B})$ in the sense that $(\mathcal{G}, \mathcal{T}) = (\mathcal{A} \cap \mathfrak{T}\mathfrak{f}, \mathcal{B} \cap \mathfrak{T})$. However, it is obvious that for a cotorsion pair $(\mathcal{A}, \mathcal{B})$ the pair $(\mathcal{A} \cap \mathfrak{T}\mathfrak{f}, \mathcal{B} \cap \mathfrak{T})$ is not always a Baer cotorsion pair since $\mathcal{B}$ does not necessarily contain all torsion cotorsion groups. For the next result we need to recall two results due to Salce [Sa] and Kulikov [Kul].

LEMMA 2.8: *Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair. Then the following are equivalent:*
(i) *$\mathcal{A}$ contains a non-trivial p-group;*
(ii) *$\mathcal{A}$ contains all p-groups;*
(iii) *every group in $\mathcal{B}$ is p-divisible.*

For an abelian group X let $t(X)$ denote its torsion subgroup.

LEMMA 2.9: *Let X, Y be two abelian groups. Then $\operatorname{Ext}(X, Y) = 0$ if and only if $\operatorname{Ext}(t(X), Y) = 0$ and $\operatorname{Ext}(X/t(X), Y) = 0$.*

In view of Lemma 2.8 and Lemma 2.9 it is reasonable to define $\pi(\mathcal{A})$ as $\pi(\mathcal{A}) = \{p \in \Pi \colon \mathbb{Z}(p) \in \mathcal{A}\}$ for a class $\mathcal{A}$ of abelian groups and to put $\pi(X) = \{p \in \Pi \colon t_p(X) \neq 0\}$ for an abelian group X. Note that

$$\pi(X) = \{p \in \Pi \colon \mathbb{Z}(p) \subseteq X\},$$

hence this definition is in accordance with the definition of $\pi(\mathcal{A})$ for classes $\mathcal{A}$ of abelian groups.

THEOREM 2.10: *Let $(\mathcal{A},\mathcal{B})$ be a cotorsion pair and let $\mathcal{G} = \mathcal{A}\cap\mathfrak{T}\mathfrak{f}$. Then $\mathcal{TC}_{\mathfrak{T}}(\mathcal{G}) = (\mathcal{B}\cap\mathfrak{T}) + (\mathfrak{C}\cap\mathfrak{T})$.*

Proof: Let $P = \pi(\mathcal{A})$. We first claim that $\mathbb{Q}^{(P)} \in \mathcal{A}$. By Lemma 2.8 it follows that $\mathbb{Z}(p^\infty) \in \mathcal{A}$ for every $p \in P$. Thus $\mathbb{Q}^{(P)}/\mathbb{Z} = \bigoplus_{p\in P}\mathbb{Z}(p^\infty) \in \mathcal{A}$. Since also $\mathbb{Z} \in \mathcal{A}$ we conclude that $\mathbb{Q}^{(P)} \in \mathcal{A}$.

Now assume that $T \in \mathfrak{T}$ satisfies $\mathrm{Ext}(G,T) = 0$ for all $G \in \mathcal{G}$. We have to prove that $T \in (\mathcal{B}\cap\mathfrak{T}) + (\mathfrak{C}\cap\mathfrak{T})$. Therefore we write T in the form $T = T'\oplus T''$ with $T' = \bigoplus_{p\notin P} T_p$ and $T'' = \bigoplus_{p\in P} T_p$. As shown above we have $\mathbb{Q}^{(P)} \in \mathcal{A}$ and hence $\mathbb{Q}^{(P)} \in \mathcal{G}$, thus $\mathrm{Ext}(\mathbb{Q}^{(P)},T) = 0$. Consequently, $\mathrm{Ext}(\mathbb{Q}^{(P)},T'') = 0$ and Proposition 2.3 implies that T'' is the direct sum of a bounded group and a divisible torsion group. Thus $T'' \in (\mathfrak{C}\cap\mathfrak{T})$.

It remains to prove that $T' \in (\mathcal{B}\cap\mathfrak{T})$. Let $M \in \mathcal{A}$; then Lemma 2.9 implies that $M/t(M) \in \mathcal{A}\cap\mathfrak{T}\mathfrak{f}$ and hence $\mathrm{Ext}(M/t(M),T') = 0$ by assumption. Let $P' = \pi(M) = \{p \in \Pi\colon t(M)_p \neq 0\}$. Then $\mathbb{Z}(p) \in \mathcal{A}$ for all $p \in P'$ and therefore $P' \subseteq P$. Again Lemma 2.8 implies that T' is p-divisible for all $p \in P$ and thus $\mathrm{Ext}(t(M),T') = 0$. This shows that $\mathrm{Ext}(M,T') = 0$ and so $T' \in (\mathcal{B}\cap\mathfrak{T})$. Hence we have $T = T'\oplus T'' \in (\mathcal{B}\cap\mathfrak{T}) + (\mathfrak{C}\cap\mathfrak{T})$, as required. ∎

However, even though Theorem 2.10 may suggest that a cotorsion pair $(\mathcal{A},\mathcal{B})$ always induces a Baer cotorsion pair, namely $(\mathcal{A}\cap\mathfrak{T}\mathfrak{f},(\mathcal{B}\cap\mathfrak{T}) + (\mathfrak{C}\cap\mathfrak{T}))$, the next proposition shows that this is, in general, not the case. Nevertheless, if $(\mathcal{A},\mathcal{B})$ is a cotorsion pair between the minimal cotorsion pair and the cotorsion pair $(\mathfrak{T}\mathfrak{f},\mathfrak{C})$ then clearly $(\mathcal{A},\mathcal{B})$ induces the minimal Baer cotorsion pair.

For the next proposition we need to recall the quasi-reduced type of a rational group $R \subseteq \mathbb{Q}$ from [StWa]. A type R is called **quasi-reduced** if $\chi_p^R(1) \in \{0,1,\infty\}$ for all $p \in \Pi$. Clearly, there exists a unique quasi-reduced type $type^{qr}(R)$ $(=R_{qr} \subseteq \mathbb{Q})$ for every type (rational group) R by putting $\chi_p^{type^{qr}(R)}(1) = 1$ if $0 < \chi_p^R(1) < \infty$ and $\chi_p^{type^{qr}(R)}(1) = \chi_p^R(1)$ otherwise.

PROPOSITION 2.11: *Let R be a rational group which is not idempotent and let $(\mathcal{A},\mathcal{B})$ be the cotorsion pair singly cogenerated by R. Then the induced pair $(\mathcal{A}\cap\mathfrak{T}\mathfrak{f},(\mathcal{B}\cap\mathfrak{T}) + (\mathfrak{C}\cap\mathfrak{T}))$ is not a Baer cotorsion pair.*

Proof: Since R is not idempotent there exists a rational group S which is not idempotent such that $\mathrm{type}(S) > \mathrm{type}(R)$ but $\mathrm{type}^{qr}(S) = \mathrm{type}^{qr}(R)$. Thus $\mathcal{TC}_{\mathfrak{T}}(S) = \mathcal{TC}_{\mathfrak{T}}(R) = \mathcal{B}\cap\mathfrak{T} = (\mathcal{B}\cap\mathfrak{T}) + (\mathfrak{T}\cap\mathfrak{C})$ by Theorem 2.10. However, by [GöShWa, Theorem 1.11] there exists an abelian group $M \in R^\perp$ such that $\mathrm{Ext}(S, M) \neq 0$. Therefore $S \notin \mathcal{A} \cap \mathfrak{T}\mathfrak{f}$ but $S \in \mathcal{FC}_{\mathfrak{Tf}}(\mathcal{TC}_{\mathfrak{T}}(R))$, showing that the induced pair $(\mathcal{A} \cap \mathfrak{Tf}, (\mathcal{B} \cap \mathfrak{T}) + (\mathfrak{T} \cap \mathfrak{C}))$ is not a Baer cotorsion pair. ∎

In contrast to the above, for cotorsion pairs generated by torsion groups we have

THEOREM 2.12: *If $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair generated by a class of torsion groups, then $(\mathcal{A} \cap \mathfrak{Tf}, (\mathcal{B} \cap \mathfrak{T}) + (\mathfrak{C} \cap \mathfrak{T}))$ is a Baer cotorsion pair.*

Proof: Let $\mathcal{O}$ be a class of torsion groups generating $(\mathcal{A}, \mathcal{B})$; then $\mathcal{O} \subseteq \mathcal{B} \cap \mathfrak{T}$. Thus, if G is a torsion-free abelian group satisfying $\mathrm{Ext}(G, T) = 0$ for all $T \in \mathcal{O}$, then $G \in \mathcal{A}$ follows and hence $G \in \mathcal{A} \cap \mathfrak{Tf}$. This shows that $(\mathcal{A} \cap \mathfrak{Tf}, \mathcal{B} \cap \mathfrak{T} + \mathfrak{C} \cap \mathfrak{T})$ is a Baer cotorsion pair. ∎

Finally, we have the following result which describes, in general, the connection between cotorsion pairs and Baer cotorsion pairs.

THEOREM 2.13: *Let $(\mathcal{A}, \mathcal{B})$ be a cotorsion pair and let $\mathcal{T} = (\mathcal{B} \cap \mathfrak{T}) + (\mathfrak{C} \cap \mathfrak{T})$. Then $(\mathcal{FC}_{\mathfrak{Tf}}(\mathcal{T}), \mathcal{T})$ is a Baer cotorsion pair.*

Proof: Follows immediately from Theorem 2.10. ∎

The pair $(\mathcal{TC}_{\mathfrak{Tf}}(\mathcal{T}), \mathcal{T})$ is called the Baer cotorsion pair naturally induced by $(\mathcal{A}, \mathcal{B})$ (see also [St3]).

3. Singly cogenerated Baer cotorsion pairs

In this section we restrict our attention to Baer cotorsion pairs which are singly cogenerated by a torsion-free abelian group G. Hence we mainly focus on determining the structure of the class $\mathcal{TC}_{\mathfrak{T}}(G)$ which, of course, gives full information about $(\mathcal{FC}_{\mathfrak{Tf}}(\mathcal{TC}_{\mathfrak{T}}(G)), \mathcal{TC}_{\mathfrak{T}}(G))$. This is closely related to Baer's Problem, the characterization of all pairs (G, T) of torsion-free abelian groups G and torsion abelian groups T satisfying $\mathrm{Ext}(G, T) = 0$. Since a full description of $\mathcal{TC}_{\mathfrak{T}}(G)$ is available for countable groups in ZFC (see [StWa]) and for arbitrary groups under $V = L$ (see [St2]) the aim of this section is to put these results into the global context of Baer cotorsion pairs. In particular, it turns out that in $V = L$ every Baer cotorsion pair is singly cogenerated by a torsion-free abelian group of size $\aleph_1$.

If $\mathcal{TC}_{\mathfrak{T}}(G)$ is maximal, i.e. $\mathcal{TC}_{\mathfrak{T}}(G) = \mathfrak{T} = \mathcal{TC}_{\mathfrak{T}}(\mathbb{Z})$, then G has to be free by Griffith's solution of the Baer problem [Gr]. On the other hand, however, if

$\mathcal{TC}_{\mathfrak{T}}(G) = \mathcal{TC}_{\mathfrak{T}}(\mathbb{Q})$ is minimal then G need not to be divisible, as the following example shows:

EXAMPLE 3.1: *Let $G = P = \prod_{n\in\omega} \mathbb{Z}$ be the Baer–Specker group. Obviously, P is not divisible; in fact, P is homogeneous of type $\mathbb{Z}$.*

However, by [GöTr, Lemma 1.3], P contains a subgroup $\mathbb{D}_\omega$ such that $\mathcal{TC}_{\mathfrak{T}}(\mathbb{D}_\omega) = \mathcal{TC}_{\mathfrak{T}}(\mathbb{Q})$ and thus $\mathcal{TC}_{\mathfrak{T}}(P) = \mathcal{TC}_{\mathfrak{T}}(\mathbb{Q})$ since $\mathbb{Q}^\perp \subseteq P^\perp \subseteq \mathbb{D}_\omega^\perp$. ∎

This example indicates that rather complicated groups may, in view of Baer cotorsion pairs, be replaced by a simpler group like $\mathbb{Q}$.

We now consider Baer cotorsion pairs singly cogenerated by a completely decomposable group C. Therefore we call a Baer cotorsion pair which is singly cogenerated by a completely decomposable group C (rational group R) a **completely decomposable Baer cotorsion pair (rational Baer cotorsion pair)**.

Definition 3.2: Let R be a rational group and $\chi^{qr}(R) = (r_p)_{p\in\Pi}$. Then $\mathfrak{T}_R$ is the class of all reduced torsion groups $T = \bigoplus_{p\in\Pi} T_p$ satisfying the following two conditions:

(i) T_p is bounded for all p such that $r_p = \infty$;

(ii) $T_p = 0$ for almost all p such that $r_p \neq 0$.

For a rational group R let $(\mathcal{G}_R, \mathcal{T}_R)$ be the rational Baer cotorsion pair singly cogenerated by R.

THEOREM 3.3: *Let R be a rational group. Then the rational Baer cotorsion pairs $(\mathcal{G}_R, \mathcal{T}_R)$ and $(\mathcal{G}_{R_{qr}}, \mathcal{T}_{R_{qr}})$ coincide and we have*

$$\mathcal{TC}_{\mathfrak{T}}(R) = \mathcal{TC}_{\mathfrak{T}}(R_{qr}) = \mathfrak{T}_R \oplus (\mathfrak{D} \cap \mathfrak{T}).$$

Moreover, for two rational groups R and S we have $(\mathcal{G}_S, \mathcal{T}_S) \leq (\mathcal{G}_R, \mathcal{T}_R)$ (or equivalently $\mathcal{TC}_{\mathfrak{T}}(R) \subseteq \mathcal{TC}_{\mathfrak{T}}(S)$) if and only if $type^{qr}(R) \geq type^{qr}(S)$.

Proof: The proof is straightforward using Proposition 2.3 and the fact that divisible (torsion) groups T always satisfy $\mathrm{Ext}(R,T) = 0$. See also [StWa]. ∎

As an immediate consequence of Theorem 3.3 we have that

$$\mathcal{TC}_{\mathfrak{T}}(R) = \mathcal{TC}_{\mathfrak{T}}(R')$$

for rational groups R, R' if and only if $type^{qr}(R) = type^{qr}(R')$. So, by what we have said before, rational Baer cotorsion pairs can be the same even for incomparable rational groups. This contrasts the analogous result from [GöShWa] for cotorsion pairs.

Before we continue note that the characterization of rational Baer cotorsion pairs immediately induces a characterization of completely decomposable Baer cotorsion pairs since $\mathcal{TC}_{\mathfrak{T}}(C) = \bigcap_{i\in I} \mathcal{TC}_{\mathfrak{T}}(R_i)$ for completely decomposable groups $C = \bigoplus_{i\in I} R_i$ with $R_i \subseteq \mathbb{Q}$, although it is not very explicit.

The key stone for characterizing singly cogenerated Baer cotorsion pairs in general is the following result from [StWa, Theorem 3.6].

THEOREM 3.4: *Let $\mathcal{T}$ be a class of torsion groups. Then $\mathcal{T} = \mathcal{TC}_{\mathfrak{T}}(C)$ for some completely decomposable group C if and only if the following conditions are satisfied:*

(i) *$\mathcal{T}$ contains all torsion cotorsion groups;*
(ii) *$\mathcal{T}$ is closed under epimorphic images;*
(iii) *$\bigoplus_{n\in\omega} \mathbb{Z}(p^n) \in \mathcal{T}$ if and only if $\mathcal{T}$ contains all p-groups for all primes p;*
(iv) *if P is an infinite set of primes, then $\bigoplus_{p\in P} \mathbb{Z}(p) \in \mathcal{T}$ if and only if $\bigoplus_{p\in P} T_p \in \mathcal{T}$ for all p-groups $T_p \in \mathcal{T}$;*
(v) *if P is an infinite set of primes such that $\bigoplus_{p\in P} \mathbb{Z}(p) \notin \mathcal{T}$ then there exists an infinite subset P' of P such that $\bigoplus_{p\in X} \mathbb{Z}(p) \notin \mathcal{T}$ for all infinite $X \subseteq P'$.*

We have the following immediate corollary.

COROLLARY 3.5: *A Baer cotorsion pair $(\mathcal{G}, \mathcal{T})$ is a completely decomposable Baer cotorsion pair if and only if $\mathcal{T}$ satisfies Theorem 3.4. In particular, a by G singly cogenerated Baer cotorsion pair is a completely decomposable Baer cotorsion pair if and only if $\mathcal{TC}_{\mathfrak{T}}(G)$ satisfies Theorem 3.4.*

We would like to note that in [St1, Theorem 4.6] the above theorem was extended by adding two more conditions in order to characterize Baer cotorsion pairs which are rational cotorsion pairs.

First examples of indecomposable torsion-free abelian groups cogenerating a completely decomposable Baer cotorsion pair are provided by the so-called Butler groups. Recall that a torsion-free abelian group G is called a B_1**-group** or **Butler group** if $\mathrm{Bext}(G, T) = 0$ for all torsion groups T where Bext consists of all balanced exact sequences; a short exact sequence $0 \to A \to B \to G \to 0$ (with G torsion-free) is called **balanced exact** if, for all rank-1 groups R, the induced homomorphism $\mathrm{Hom}(R, B) \to \mathrm{Hom}(R, G)$ is surjective. Note that almost completely decomposable groups (that means torsion-free abelian groups containing a completely decomposable subgroup of finite index) and finite rank Butler groups (that means pure subgroups of finite rank completely decomposable groups) are B_1-groups. We have the following direct consequence of [StWa, Theorem 4.6]:

THEOREM 3.6: *Let G be a B_1-group. Then $\mathcal{TC}_{\mathfrak{T}}(G) = \mathcal{TC}_{\mathfrak{T}}(\bigoplus_{\tau\in Tst(G)} R_\tau)$. Hence the Baer cotorsion pair cogenerated by G is a completely decomposable Baer cotorsion pair and completely determined by the typeset of G.*

As an immediate consequence we obtain

COROLLARY 3.7: *Every Baer cotorsion pair cogenerated by a class of B_1-groups is a completely decomposable Baer cotorsion pair.*

However, it was shown in [St1] that there are also non-Butler groups G having the property stated in Theorem 3.6 and that the class of abelian groups satisfying Theorem 3.6 is neither closed under extensions nor direct summands.

We now turn our attention to countable groups and start with the finite rank case. In [St1, Theorem 2.4] the following was proved.

THEOREM 3.8: *Let G be a torsion-free abelian group of finite rank. Then $\mathcal{TC}_{\mathfrak{T}}(G) = \mathcal{TC}_{\mathfrak{T}}(R)$ for some rational group R.*

It was also proved in [St1, Theorem 2.5] that R in Theorem 3.8 can be chosen to be the outer type $\mathrm{OT}(G)$ of G (for definition see [Ar, Chapter 3]) and a result due to Warfield [Wa] allows one to determine the outer type, and hence $\mathcal{TC}_{\mathfrak{T}}(G)$, explicitly for a torsion-free abelian group G of finite rank. The following theorem is now immediate.

THEOREM 3.9: *Every Baer cotorsion pair which is singly cogenerated by a torsion-free abelian group of finite rank is a rational Baer cotorsion pair.*

We would like to remark that it was shown in [St1] that for almost all types R, the class $\mathcal{TC}_{\mathfrak{T}}(R)$ can be realized as $\mathcal{TC}_{\mathfrak{T}}(G)$ for an indecomposable, almost decomposable group of rank n for any natural number n. This proves that the structure of the group G is less effected by $\mathcal{TC}_{\mathfrak{T}}(G)$ than, for example, by $G^{\perp}$, even for finite rank abelian groups. Thus the Baer cotorsion pair cogenerated by G does not give much information about the group G itself.

As a corollary we obtain a general version of Griffith's solution of the Baer problem for abelian groups of finite rank.

COROLLARY 3.10: *Let G be a torsion-free abelian group of finite rank and homogeneous of idempotent type R. Then G is completely decomposable if and only if the Baer cotorsion pairs cogenerated by G and R coincide, i.e. if $\mathcal{TC}_{\mathfrak{T}}(G) = \mathcal{TC}_{\mathfrak{T}}(R)$.*

Proof: One implication is trivial, hence we assume $\mathcal{TC}_{\mathfrak{T}}(G) = \mathcal{TC}_{\mathfrak{T}}(R)$. Since $\mathcal{TC}_{\mathfrak{T}}(G) = \mathcal{TC}_{\mathfrak{T}}(\mathrm{OT}(G))$ we obtain, by Theorem 3.3, that the quasi-reduced types of R and $\mathrm{OT}(G)$ coincide. However, R is idempotent and thus the types R and $\mathrm{OT}(G)$ are equal. Hence $\mathrm{IT}(G) = R = \mathrm{OT}(G)$ and so the result follows by [Ar, Proposition 3.1.13]. ∎

COROLLARY 3.11: *Let G be a torsion-free abelian group of finite rank. Then G is free if and only if the Baer cotorsion pair cogenerated by G is the maximal one, i.e. $\mathcal{TC}_{\mathfrak{T}}(G) = \mathcal{TC}_{\mathfrak{T}}(\mathbb{Z})$.*

Proof: Again one implication is trivial, hence we assume $\mathcal{TC}_{\mathfrak{T}}(G) = \mathcal{TC}_{\mathfrak{T}}(\mathbb{Z})$. We want to apply Corollary 3.10. Therefore we need that G is homogeneous of type $\mathbb{Z}$. But $\mathcal{TC}_{\mathfrak{T}}(G) = \mathcal{TC}_{\mathfrak{T}}(\mathbb{Z})$ implies that $\mathrm{OT}(G) = \mathbb{Z}$ and so G must be homogeneous of type $\mathbb{Z}$. ∎

Let us note that Corollary 3.10 may fail if we do not assume that R is idempotent (see [St1, Lemma 2.10]), hence Griffith's solution of the Baer problem cannot be generalized to homogeneous groups of non-idempotent type.

THEOREM 3.12: *Every Baer cotorsion pair which is cogenerated by a class of countable torsion-free abelian groups is singly cogenerated by a completely decomposable group.*

Proof: By [StWa] $\mathcal{TC}_{\mathfrak{T}}(C)$ satsifies the conditions in Theorem 3.12 for any countable group C. Hence the theorem is immediate. ∎

Let us remark that the above Theorem 3.12 cannot be strengthened to rational Baer cotorsion pairs, not even for completely decomposable Baer cotorsion pairs, as was shown in [StWa, Lemma 4.2].

We now focus on the main result of this section which states that under the assumption of $V = L$, every singly cogenerated Baer cotorsion pair is singly cogenerated by the direct sum of an almost free abelian group of cardinality $\aleph_1$ and a countable completely decomposable group. Since the class of abelian groups of cardinality $\aleph_1$ is a set of size at most $2^{\aleph_1}$ we also obtain that every Baer cotorsion pair is singly cogenerated by an abelian group of cardinality at most $\aleph_1$ in $V = L$. Moreover, large classes of Baer cotorsion pairs are already completely decomposable Baer cotorsion pairs (in Gödel's universe).

In [St1, Theorem 2.7] it was shown that condition Theorem 3.4 (iv) always holds under the assumption of $V = L$ for any class $\mathcal{TC}_{\mathfrak{T}}(G)$ with G a torsion-free abelian group.

THEOREM 3.13 ($V = L$): *Let G be a torsion-free abelian group. If P is an infinite set of primes, then $\bigoplus_{p\in P}\mathbb{Z}(p) \in \mathcal{TC}_{\mathfrak{T}}(G)$ if and only if $\bigoplus_{p\in P} T_p \in \mathcal{TC}_{\mathfrak{T}}(G)$ for all p-groups $T_p \in \mathcal{TC}_{\mathfrak{T}}(G)$. In particular, Theorem 3.4 (iv) holds for $\mathcal{TC}_{\mathfrak{T}}(G)$.*

Note that Theorem 3.13 already implies that, for a large class of torsion-free abelian groups G, there exists a completely decomposable group C such that $\mathcal{TC}_{\mathfrak{T}}(G) = \mathcal{TC}_{\mathfrak{T}}(C)$ if we assume $V = L$. In fact, this is true for all torsion-free abelian groups G satisfying Theorem 3.4 (v). Thus, in Gödel's universe, many singly cogenerated Baer cotorsion pairs are completely decomposable Baer cotorsion pairs.

In [ShSt] Shelah and the author were finally able to prove the following results which give a full characterization of $\mathcal{TC}_{\mathfrak{T}}(G)$ for torsion-free abelian groups G. However, the proof was not absolutely correct and was corrected in [St4]. Therefore we state the main results of [ShSt] in the corrected version and give the main steps of the proof if it is different from the proof given in [ShSt].

For an infinite subset $P \subseteq \Pi$ we define $T_P = \bigoplus_{p\in P}\mathbb{Z}(p)$ where $\mathbb{Z}(p)$ denotes the cyclic group of order p. Recall that, for an infinite set I, an **ideal** on I is a subset D of $\mathcal{P}(I)$ such that

(i) $X, Y \in D$ implies $X \cup Y \in D$;

(ii) $X \subseteq Y \subseteq I$ with $Y \in D$ implies $X \in D$;

(iii) $\emptyset \in D$ but $I \notin D$.

For a torsion-free abelian group G it is not hard to see that the set

$$D = \{P \subseteq \Pi: T_P \in \mathcal{TC}(G)\}$$

forms an ideal on $\mathcal{P}(\Pi)$ containing all finite subsets of Π. In [ShSt] it was shown that every such ideal may appear. It was even claimed that one can choose any ideal on $\mathcal{P}(\bar{\Pi})$ where $\bar{\Pi} = \{p^n: p \in \Pi, n \in \omega\}$. This is not correct, but all the proofs remain correct if one replaces $\bar{\Pi}$ by Π (see also [St4]). To avoid additional notation let us allow an ideal in $\mathcal{P}(\Pi)$ to contain Π itself. The following is [ShSt, Theorem 2.6].

THEOREM 3.14 (CH): *Let $I \subseteq \mathcal{P}(\Pi)$ be an ideal containing all finite subsets of Π. Then there exists an $\aleph_1$-free group G of cardinality $\aleph_1$ such that, for every $P \subseteq \Pi$, $T_P \in \mathcal{TC}(G)$ if and only if $P \in I$.*

As a corollary we obtain the following (corrected) result [ShSt, Corollary 3.2].

THEOREM 3.15 ($V = L$): *For every abelian torsion-free group G there exists an abelian group H of cardinality $\aleph_1$ such that $\mathcal{TC}_{\mathfrak{T}}(G) = \mathcal{TC}_{\mathfrak{T}}(H)$. Hence every Baer cotorsion pair is singly cogenerated by an abelian group of cardinality $\aleph_1$.*

Proof: Let G be given and put $I = \{P \subseteq \Pi\colon \bigoplus_{p\in P}\mathbb{Z}(p) \in \mathcal{TC}_{\mathfrak{T}}(G)\}$. Then it is easy to see that I is an ideal on $\mathcal{P}(\Pi)$ containing all finite subsets of Π. Thus, by Theorem 3.14, there exists an $\aleph_1$-free group G' of cardinality $\aleph_1$ such that, for every subset $P \subseteq \Pi$, $\bigoplus_{p\in P}\mathbb{Z}(p) \in \mathcal{TC}(G')$ if and only if $P \in I$. Let $Q = \{p \in \Pi\colon \bigoplus_{n<\omega}\mathbb{Z}(p^n) \notin \mathcal{TC}_{\mathfrak{T}}(G)\}$ and put $H = G' \oplus \bigoplus_{p\in Q}\mathbb{Q}^{(p)}$. It now follows easily as in [ShSt] that H is as required.

Since all abelian groups of size $\aleph_1$ form a set of cardinality at most $2^{\aleph_1}$ it is immediate that every Baer cotorsion pair is cogenerated by a set of torsion-free abelian groups and hence by a single group which again can be replaced by an abelian group of size $\aleph_1$. ∎

However, this is not a result provable in ZFC. Already in the local case, it is undecidable in ZFC whether or not any singly cogenerated Baer cotorsion pair is a completely decomposable cotorsion pair (see [StWa, Lemma 3.10 and Proposition 3.11]).

4. Singly generated Baer cotorsion pairs

In this section we study singly generated Baer cotorsion pairs. In particular, we want to determine when a Baer cotorsion pair singly generated by a torsion group is a rational Baer cotorsion pair. Our first theorem gives a necessary condition which, at least in Gödel's constructible universe, will be shown to be sufficient as well. Recall that $\pi(T)$ is the set of all primes for which the torsion group T has a non-trivial p-component. We decompose $\pi(T)$ into two subsets by putting $\pi_b(T) = \{p \in \pi(T)\colon T_p \text{ is bounded}\}$ and $\pi_{ub}(T) = \{p \in \pi(T)\colon T_p \text{ is unbounded}\}$. Finally, let $\pi_0(T) = \{p \in \Pi\colon T_p = 0\}$.

THEOREM 4.1: *Let T be a torsion group. If the Baer cotorsion pair singly generated by T is a rational Baer cotorsion pair cogenerated by $R \subseteq \mathbb{Q}$, then T has only finitely many non-trivial bounded primary components and R is idempotent.*

Proof: Let T be given and assume that the Baer cotorsion pair singly generated by T is the rational Baer cotorsion pair cogenerated by $R \subseteq \mathbb{Q}$ and so, in particular, $\operatorname{Ext}(R,T) = 0$. Thus

$$\mathcal{FC}_{\mathfrak{T}\mathfrak{f}}(T) = \mathcal{FC}_{\mathfrak{T}\mathfrak{f}}(\mathcal{TC}_{\mathfrak{T}}(R)) \quad \text{and} \quad \mathcal{TC}_{\mathfrak{T}}(\mathcal{FC}_{\mathfrak{T}\mathfrak{f}}(T)) = \mathcal{TC}_{\mathfrak{T}}(R).$$

Assume that $\pi_b(T)$ is infinite. By Proposition 2.3, $\mathbb{Q}^{(p)}=\langle 1/p^n : n\in\omega\rangle\in\mathcal{FC}_{\mathfrak{T}\mathfrak{f}}(T)$ for every prime $p\in\pi_b(T)$. Moreover, if $\chi(R)=(r_p)_{p\in\Pi}$ then Proposition 2.3 implies $r_p=0$ for almost all primes $p\in\pi_b(T)$. Hence

$$\pi_b'(T)=\{p\in\pi_b(T)\colon r_p=0\}$$

is infinite. Fix $p\in\pi_b'(T)$ and an unbounded reduced p-group T'. Since $p\in\pi_b'(T)$ it follows that $T'\in\mathcal{TC}_{\mathfrak{T}}(R)$. On the other hand, however, $\mathbb{Q}^{(p)}\in\mathcal{FC}_{\mathfrak{T}\mathfrak{f}}(T)$ implies that $T'\notin\mathcal{TC}_{\mathfrak{T}}(\mathcal{FC}_{\mathfrak{T}\mathfrak{f}}(T))=\mathcal{TC}_{\mathfrak{T}}(R)$ — a contradiction. Therefore $\pi_b(T)$ is finite.

It remains to show that R is idempotent. Seeking a contradiction assume that R is not idempotent and hence $\sigma(R)=\{p\in\Pi\colon 0<r_p<\infty\}$ is infinite. Then T_p must be trivial or divisible for almost all primes $p\in\sigma(R)$ by Proposition 2.3. As above, there exists a prime p such that $\mathbb{Q}^{(p)}\in\mathcal{FC}_{\mathfrak{T}\mathfrak{f}}(T)$ and $T'\in\mathcal{TC}_{\mathfrak{T}}(R)$ for some unbounded reduced p-group T' yielding a contradiction to $\mathcal{TC}_{\mathfrak{T}}(\mathcal{FC}_{\mathfrak{T}\mathfrak{f}}(T))=\mathcal{TC}_{\mathfrak{T}}(R)$. This completes the proof. ∎

An interesting, still open question is the following

QUESTION 4.2: *If T has infinitely many non-trivial bounded primary components, can the Baer cotorsion pair generated by T be a completely decomposable Baer cotorsion pair?*

In order to prove some sort of converse to Theorem 4.1 we need the following immediate consequence of [St2, Theorem 3.9]. For an abelian group G let $\mathrm{Subgr}(G)=\{U\in\mathfrak{Ab}\colon U\subseteq\bigoplus_\lambda G \text{ for some cardinal } \lambda\}$.

THEOREM 4.3 ($V=L$): *Let T be a torsion group. Then there exists an idempotent rational group R such that $\mathcal{FC}_{\mathfrak{T}\mathfrak{f}}(T)=\mathrm{Subgr}(R)$ if and only if T has only finitely many non-trivial bounded primary components. In this case, the Baer cotorsion pair singly generated by T equals the rational Baer cotorsion pair singly cogenerated by R.*

Proof: In [St2, Theorem 3.9] it was shown that under the assumption of $V=L$ there exists an idempotent rational group R such that $\bigoplus_\lambda R$ is λ-universal for T for any cardinal λ. Since λ-universal for T means that for any cardinal λ, any torsion-free abelian group G of rank less than or equal to λ which satisfies $\mathrm{Ext}(G,T)=0$ can be embedded into $\bigoplus_\lambda R$, it follows immediately that $\mathcal{FC}_{\mathfrak{T}\mathfrak{f}}(T)=\mathrm{Subgr}(R)$. For finite λ this was even proved to be an if and only if statement in ZFC in [St2, Theorem 3.3]. Hence the claim follows. ∎

Using Theorem 4.1 we obtain

COROLLARY 4.4 ($V = L$): *Let T be a torsion group. The Baer cotorsion pair singly generated by T is a rational Baer cotorsion pair (generated by an idempotent rational group) if and only if T has only finitely many non-trivial bounded primary components.*

5. The lattice of Baer cotorsion pairs

In this section we consider the lattice of Baer cotorsion pairs. From what we have proved in the last sections it will turn out that in Gödel's universe the lattice of all Baer cotorsion pairs is almost in bijection with the lattice of ideals of $\mathcal{P}(\Pi)$ containing all finite subsets of Π. Thus a generalization of [GöShWa, Theorem 3.1] is not possible. However, we shall give some results about embedding posets in the lattice of rational and completely decomposable Baer cotorsion pairs respectively. Moreover, we try to characterize those ideals in $\mathcal{P}(\Pi)$ which come from completely decomposable Baer cotorsion pairs.

We first consider the lattice $(\mathfrak{L}_{rat}, \leq)$ of all rational Baer cotorsion pairs. In [GöShWa, Theorem 3.1] it was shown that any poset can be embedded into the lattice of all cotorsion pairs, in fact even into the lattice of all cotorsion pairs which are singly cogenerated by a torsion-free abelian group. It is our aim to prove a similar result for (rational, completely decomposable) Baer cotorsion pairs. The first proposition is an analogue to [GöShWa, Theorem 1.11]. Recall that a rational group $R \subseteq \mathbb{Q}$ is called **quasi-reduced** if $\chi_p^R(1) \in \{0, 1, \infty\}$ for all primes p except for finitely many. Naturally, the quasi-reduced types form a lattice with the natural ordering $\leq$ which we will denote by $(\mathrm{type}^{qr}, \leq)$.

PROPOSITION 5.1: *The lattice of quasi-reduced types $(Type_{qr}, \leq)$ is anti-isomorphic to the lattice of rational Baer cotorsion pairs via the mapping* $\mathrm{type}^{qr}(R) \mapsto (\mathcal{FC}_{\mathfrak{T}\mathfrak{f}}(\mathcal{TC}_{\mathfrak{T}}(R)), \mathcal{TC}_{\mathfrak{T}}(R))$.

Proof: The proof follows immediately from Theorem 3.3. Clearly, the mapping $\mathrm{type}^{qr}(R) \mapsto (\mathcal{FC}_{\mathfrak{T}\mathfrak{f}}(\mathcal{TC}_{\mathfrak{T}}(R)), \mathcal{TC}_{\mathfrak{T}}(R))$ is order-reversing and a monomorphism by Theorem 3.3. To show surjectivity, let R be any type and S its quasi-reduced type. Again Theorem 3.3 shows that $\mathcal{TC}_{\mathfrak{T}}(R) = \mathcal{TC}_{\mathfrak{T}}(S)$ and hence the two Baer cotorsion pairs singly cogenerated by R and S coincide. ∎

THEOREM 5.2: *Let I be a countable set. Then the power set $\mathcal{P} = \mathcal{P}(I)$ can be embedded into the lattice $(\mathfrak{L}_{rat}, \leq)$ of rational Baer cotorsion pairs.*

Proof: We identify I with the set of natural primes Π, hence it is enough to show that $\mathcal{P}(\Pi)$ can be embedded into $(\mathfrak{L}_{rat}, \leq)$. Let $X \in \mathcal{P}(\Pi)$ and put

$R_X = \mathbb{Q}^{(X)}$. Then the mapping $(\mathcal{P}(\Pi), \subseteq) \to (\mathfrak{L}_{rat}, \leq)$ sending X onto R_X is an order-reversing embedding of $(\mathcal{P}(\Pi), \subseteq)$ into $(\mathfrak{L}_{rat}, \leq)$. Since the mapping $(\mathcal{P}, \subseteq) \to (\mathcal{P}, \subseteq)$ $(X \mapsto I \setminus X)$ is an order-reversing isomorphism we are done. ∎

By results due to Baer (see [Ba2] and Eda (see [Ed]) there exists an antichain of length $2^{\aleph_0}$ in $(\mathfrak{L}_{rat}, \leq)$ as well as a descending chain of uncountable length. However, the exact length of a maximal descending chain in $(\mathfrak{L}_{rat}, \leq)$ depends on the underlying set theory.

We now study the lattice $(\mathfrak{L}_{cd}, \leq)$ of all completely decomposable Baer cotorsion pairs.

THEOREM 5.3: *Let I be a set of cardinality less than or equal to $2^{\aleph_0}$. Then the power set $(\mathcal{P}(I), \subseteq)$ can be embedded into the lattice $(\mathfrak{L}_{cd}, \leq)$ of all completely decomposable Baer cotorsion pairs.*

Proof: We divide the set of natural primes Π into $2^{\aleph_0}$ almost disjoint subsets Π_i $(i \in 2^{\aleph_0})$. Put $M = \{\Pi_i \colon i \in 2^{\aleph_0}\}$. Without loss of generality we may assume that $|I| = 2^{\aleph_0}$, and we now identify I with M and it is obviously enough to embed the power set $\mathcal{P}(M)$ into $(\mathfrak{L}_{cd}, \leq)$. Since the mapping $(\mathcal{P}(M), \subseteq)$ $\to (\mathcal{P}(M), \subseteq)$ $(X \mapsto I \setminus X)$ is an order-reversing isomorphism, we are done if we can find an order-reversing monomorphism of $(\mathcal{P}(M), \subseteq)$ into $(\mathfrak{L}_{cd}, \leq)$. For $P \in M$ we let $R_P = \langle 1/p : p \in P \rangle \subseteq \mathbb{Q}$ and for $X \in \mathcal{P}(M)$ we put $C_X = \bigoplus_{P \in X} R_P$. Then each C_X is completely decomposable and the typeset of C_X is the meet closure of the types R_P $(P \in X)$. Clearly, if $X \subseteq Y$, then $C_X \subseteq C_Y$ and hence $\mathcal{TC}_{\mathfrak{T}}(C_Y) \subseteq \mathcal{TC}_{\mathfrak{T}}(C_X)$. Conversely, assume that $\mathcal{TC}_{\mathfrak{T}}(C_Y) \subseteq \mathcal{TC}_{\mathfrak{T}}(C_X)$ for some $X, Y \in \mathcal{P}(M)$ and assume that $X \not\subseteq Y$. Then there exists $P \in X \setminus Y$. Put $T = \bigoplus_{p \in P} \mathbb{Z}(p)$. Then $T \notin \mathcal{TC}_{\mathfrak{T}}(C_X)$ since $\chi_p^{R_P}(1) \neq 0$ for all $p \in P$. Now choose $Q \in Y$; then Q and P are almost disjoint, hence $T \in \mathcal{TC}_{\mathfrak{T}}(R_Q)$ and therefore $T \in \mathcal{TC}_{\mathfrak{T}}(C_Y)$. Thus $\mathcal{TC}_{\mathfrak{T}}(C_Y) \not\subseteq \mathcal{TC}_{\mathfrak{T}}(C_X)$ — a contradiction. Thus the mapping $X \mapsto (\mathcal{FC}_{\mathfrak{T}\mathfrak{f}}(\mathcal{TC}_{\mathfrak{T}}(C_X)), \mathcal{TC}_{\mathfrak{T}}(C_X))$ is an order-reversing monomorphism from $(\mathcal{P}(M), \subseteq)$ into $(\mathfrak{L}_{cd}, \leq)$. ∎

COROLLARY 5.4: *Any poset $(X, \leq)$ of cardinality less than or equal to $2^{\aleph_0}$ can be embedded into the lattice of all completely decomposable Baer cotorsion pairs.*

Proof: It is known that any poset $(X, \leq)$ of size λ can be embedded into a power set of cardinality 2^{λ}, hence the result follows from Theorem 5.3. ∎

Note that two completely decomposable groups with the same typeset necessarily have the same $\mathcal{TC}_{\mathfrak{T}}$-class, hence cogenerate the same Baer cotorsion pair. Since there are only $2^{\aleph_0}$ types and hence $2^{2^{\aleph_0}}$ possible typesets, it follows that Theorem 5.3 is as best as possible. Moreover, since any ordinal which is embeddable into the power set of μ has length at most μ, we cannot expect to find embeddings of all posets of cardinality $2^{2^{\aleph_0}}$ into the lattice of all completely decomposable Baer cotorsion pairs in general.

Finally, we turn to the lattice of all Baer cotorsion pairs assuming $V = L$. Let $\mathfrak{I}$ be the lattice of ideals of $\mathcal{P}(\Pi)$ containing all finite subsets of $\mathcal{P}(\Pi)$. We define on $\mathfrak{I} \times \mathcal{P}(\Pi)$ a partial order by letting $(I, P) \leq (I', P')$ if and only if $I \subseteq I'$ and $P' \subseteq P$.

THEOREM 5.5 ($V = L$): *There is an order-reversing bijection σ between the lattice of all Baer cotorsion pairs and $\mathfrak{I} \times \mathcal{P}(\Pi)$.*

Proof: Let $(\mathcal{G}, \mathcal{T})$ be a Baer cotorsion pair and put

$$I = \{P \subseteq \mathcal{P}(\Pi) \colon \bigoplus_{p \in P} \mathbb{Z}(p) \in \mathcal{T}\} \quad \text{and} \quad Q = \{p \in \Pi \colon \bigoplus_{n<\omega} \mathbb{Z}(p^n) \notin \mathcal{T}\}.$$

Then it is immediate that $I \in \mathfrak{I}$ and thus the mapping $(\mathcal{G}, \mathcal{T}) \mapsto (I, Q)$ is an order-reversing function from the lattice of all Baer cotorsion pairs to $\mathfrak{I} \times \mathcal{P}(\Pi)$. By Theorem 3.14 and Theorem 3.15 this mapping is a bijection. ∎

The lattice $\mathfrak{I}$ is even more complicated than the lattice of types and hence very likely its structure depends on the underlying set theory. It seems hopeless to characterize all partially ordered sets which can be embedded into $\mathfrak{I}$. However, we shall give some results on the ideals in $\mathfrak{I}$ that come from a completely decomposable Baer cotorsion pair.

Since the ideals in $\mathfrak{I}$ contain all finite subsets of Π it is reasonable to look at the Boolean algebra $(\mathcal{P}(\Pi)/Fin(\Pi), \leq)$ where $Fin(\Pi)$ denotes the set of all finite subsets of Π. Here $[P] \leq [P']$ is satisfied for two cosets $[P], [P'] \in \mathcal{P}(\Pi)/Fin(\Pi)$ if and only if $P' \backslash P$ is finite. Note that $\mathcal{P}(\Pi)/Fin(\Pi)$ is an atomless Boolean algebra which satisfies the strong countable separation property by [Ko1, Example 5.28]. Thus [Ko1, Proposition 5.29] implies that $\mathcal{P}(\Pi)/Fin(\Pi)$ is ω_1-universal, which means that every Boolean algebra of size at most $\aleph_1$ is embeddable into $\mathcal{P}(\Pi)/Fin(\Pi)$.

Let $\rho \colon \mathcal{P}(\Pi) \to \mathcal{P}(\Pi)/Fin(\Pi)$ be the canonical epimorphism. Then ρ induces an order-preserving bijection $\bar{\rho}$ between $\mathfrak{I}$ and the lattice of ideals in $\mathcal{P}(\Pi)/Fin(\Pi)$ given by $I \mapsto \{\rho(P) \colon P \in I\} = \bar{I}$ (see [Ko1, Exercise 6, page 84]).

PROPOSITION 5.6: *A Baer cotorsion pair is a completely decomposable Baer cotorsion pair if and only if the corresponding ideal $\bar{I}$ satisfies the following condition:*

$$\text{for every } \bar{P} \notin \bar{I} \text{ there exists } \overline{P'} \leq \bar{P} \text{ such that } \overline{P''} \notin \bar{I} \text{ for every } 0 \neq \overline{P''} \leq \overline{P'}.$$

Proof: The proof follows from Theorem 3.4. By Theorem 3.4 condition (v) the ideal I corresponds to a completely decomposable Baer cotorsion pair if and only if for every $P \notin I$ there exists an infinite subset $P' \subseteq P$ such that $P'' \notin I$ for all infinite subsets $P'' \subseteq P'$. Passing to the Boolean algebra $\mathcal{P}(\Pi)/Fin(\Pi)$ proves the proposition. ∎

We have an easy lemma.

LEMMA 5.7: *If $\bar{I}$ is an ideal on $\mathcal{P}(\Pi)/Fin(\Pi)$ which is singly generated, then the corresponding Baer cotorsion pair is a completely decomposable Baer cotorsion pair.*

Proof: All one has to do is to check the condition from Proposition 5.6 which is easily established. ∎

So far there is no complete characterization available for the ideals in $\mathcal{P}(\Pi)/Fin(\Pi)$. Moreover, the lattice structure of the ideals of $\mathcal{P}(\Pi)/Fin(\Pi)$ is not known. Thus, we pose the following open question:

QUESTION 5.8: *What posets can be embedded into the lattice of all Baer cotorsion pairs? What ideals $\bar{I}$ correspond to completely decomposable Baer cotorsion pairs ? Characterize the ideal lattice of $\mathcal{P}(\Pi)/Fin(\Pi)$.*

References

[Ar] D. Arnold, *Torsion-free abelian groups and finite dimensional $\mathbb{Q}$-algebras*, Lecture Notes in Mathematics **931**, Springer-Verlag, New York, 1982.

[Ba1] R. Baer, *The subgroup of the elements of finite order of an abelian group*, Annals of Mathematics **37** (1936), 766–781.

[Ba2] R. Baer, *Abelian groups without elements of finite order*, Duke Mathematical Journal **3** (1937), 68–122.

[Baz1] S. Bazzoni, *Cotilting modules are pure-injective*, Proceedings of the American Mathematical Society **131** (2003), 3665–3672.

[Baz2] S. Bazzoni, *n-cotilting and n-tilting modules,* Journal of Algebra **273** (2004), 359–372.

[Baz3] S. Bazzoni, *n-cotilting modules and pure injectivity,* The Bulletin of the London Mathematical Society **36** (2004), 599–612.

[BazEkTr] S. Bazzoni, P. C. Eklof and J. Trlifaj, *Tilting cotorsion pairs,* preprint.

[BazSa] S. Bazzoni and L. Salce, *An independence result on cotorsion theories over valuation domains,* Journal of Algebra **243** (2001), 294–320.

[BiBaEn] L. Bican, R. El Bashir and E. Enochs, *All modules have flat covers,* The Bulletin of the London Mathematical Society **33** (2001), 385–390.

[Di] S. E. Dickson, *A torsion theory for abelian categories,* Transactions of the American Mathematical Society **121** (1966), 223–235.

[Ed] K. Eda, *A characterization of $\aleph_1$-free abelian groups and its application to the Chase radical,* Mathematische Annalen **261** (1982), 359–385.

[EkFuSh] P. C. Eklof, L. Fuchs and S. Shelah, *Baer modules over domains,* Transactions of the American Mathematical Society **322** (1990), 547–560.

[EkHu] P. C. Eklof and M. Huber, *Abelian group extensions and the axiom of constructibility,* Commentarii Mathematici Helvetici **54** (1979), 440–457.

[EkMe] P. C. Eklof and A. Mekler, *Almost Free Modules, Set-Theoretic Methods,* Revised Edition, North-Holland Mathematical Library, Amsterdam, New York, 2002.

[EkShTr] P. C. Eklof, S. Shelah and J. Trlifaj, *On the cogeneration of cotorsion pairs,* Journal of Algebra **277** (2004), 572–576.

[EkTr2] P. C. Eklof and J. Trlifaj, *Covers induced by* Ext, Journal of Algebra **231** (2000), 640–651.

[Fu1] L. Fuchs, *Infinite Abelian Groups — Volume I,* Academic Press, New York–London, 1970.

[Fu2] L. Fuchs, *Infinite Abelian Groups — Volume II,* Academic Press, New York–London, 1973.

[GöSh1] R. Göbel and S. Shelah, *Almost free splitters,* Colloquium Mathematicum **81** (1999), 193–221.

[GöSh2] R. Göbel and S. Shelah, *Cotorsion theories and splitters,* Transactions of the American Mathematical Society **352** (2000), 5357–5379.

[GöSh3] R. Göbel and S. Shelah, *An addendum and corrigendum to: Almost free splitters,* Colloquium Mathematicum **88** (2001), 155–158.

[GöShWa] R. Göbel, S. Shelah and S. L. Wallutis, *On the lattice of cotorsion theories,* Journal of Algebra **238** (2001), 292–313.

[GöTr] R. Göbel and J. Trlifaj, *Cotilting and a hierachy of almost cotorsion groups*, Journal of Algebra **224** (2000), 110–122.

[GöTr3] R. Göbel and J. Trlifaj, *Approximations and Endomorphism Algebras of Modules*, de Gruyter, Berlin, to appear.

[Goe] P. Göters, *Generating cotorsion theories and injective classes*, Acta Mathematica Hungarica **51** (1988), 99–107.

[Gr] P. Griffith, *A solution to the splitting mixed group problem of Baer*, Transactions of the American Mathematical Society **139** (1969), 261–269.

[Ha] J. Hausen, *Automorphismen gesättigte Klassen abzählbarer abelscher Gruppen*, Studies on Abelian Groups, Springer, Berlin, 1968, pp. 147–181.

[Ko1] S. Koppelberg, *Handbook of Boolean Algebras I*, North-Holland, Amsterdam, 1988.

[Kul] Y. A. Kulikov, *Conditions under which the group of abelian extensions is zero* (in Russian), Moskovskii Gosudarstvennyi Pedagogicheskii Institut im. V. I. Lenina U Chenye Zapishi **375** (1971), 41–55.

[Ma] A. Mader, *Almost completely decomposable abelian groups*, Algebra, Logic and Applications **13**, Gordon and Breach, London, 2000.

[Sa] L. Salce, *Cotorsion theories for abelian groups*, Symposia Mathematica **23** (1979), 11–32.

[ShSt] S. Shelah and L. Strüngmann, *Cotorsion theories cogenerated by $\aleph_1$-free abelian groups*, Rocky Mountain Journal of Mathematics **32** (2002), 1617–1626.

[St1] L. Strüngmann, *Torsion groups in cotorsion theories*, Rendiconti del Seminario Matematico dell'Università di Padova **107** (2002), 35–55.

[St2] L. Strüngmann, *On problems by Baer and Kulikov using $V = L$*, Illinois Journal of Mathematics **46** (2002), 477–490.

[St3] L. Strüngmann, $(\mathfrak{V}, \mathfrak{W})$*-cotorsion pairs*, submitted.

[St4] L. Strüngmann, *Derived cotorsion theories*, Habilitationsschrift, Universität Duisburg-Essen, Germany, 2004.

[StWa] L. Strüngmann and S. L. Wallutis, *On the torsion groups in cotorsion classes*, AGRAM 2000 Conference, Perth, Western Australia, Contemporary Mathematics **273** (2001), 269–283.

[Tr1] J. Trlifaj, *Infinite dimensional tilting modules and cotorsion pairs*, to appear.

[Tr2] J. Trlifaj, *Cotorsion theories induced by tilting and cotilting modules*, AGRAM 2000 Conference, Perth, Western Australia, Contemporary Mathematics **273** (2001), 343–355.

[Wa] R.B. Warfield Jr., *Homomorphisms and duality for torsion-free groups,* Mathematische Zeitschrift **107** (1968), 189–200.

[Wi] W. J. Wickless, *Projective classes of torsion-free abelian groups II,* Acta Mathematica Hungarica **44** (1984), 13–20.

[Xu] J. Xu, *Flat covers of modules,* Lecture Notes in Mathematics **1634**, Springer-Verlag, Berlin, 1996.